EXPANSION OF A GAS IN A CLOSED SYSTEM

When a gas is expanded in a cylinder *Figure 2* the pressure falls and the volume increases as the piston is pushed outward by the energy in the gas.

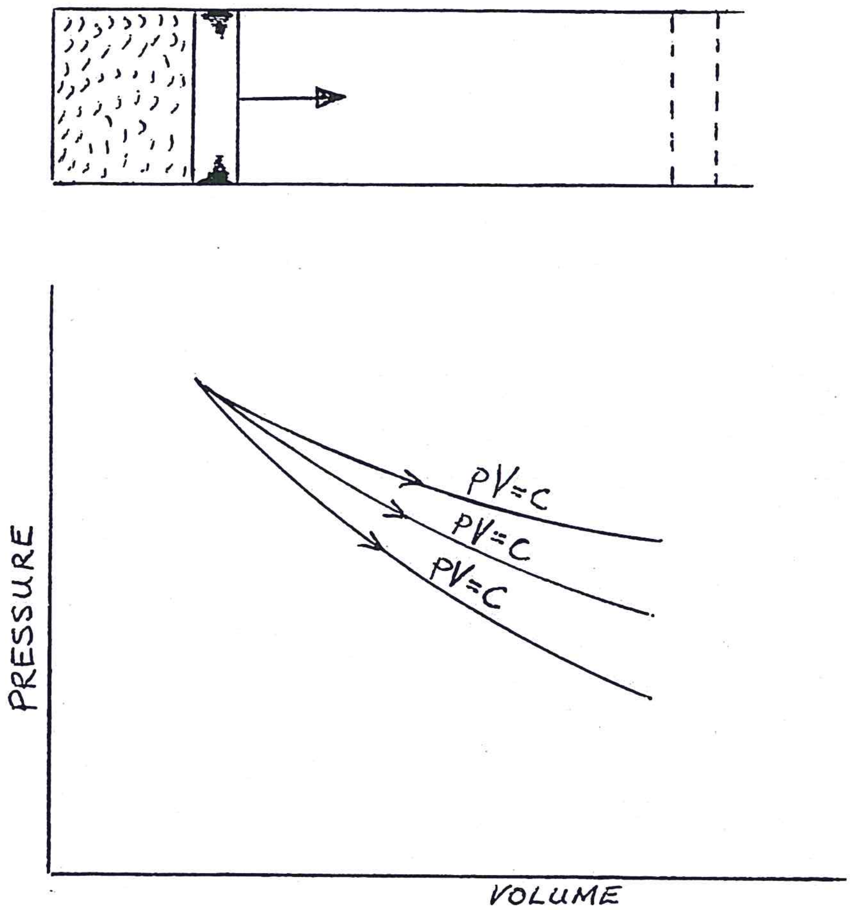

Figure 2

This is exactly the opposite to compression. Work is done by the gas in pushing the piston outward and there is a tendency for the temperature to fall due to the heat energy in the gas being converted into mechanical energy. Therefore to expand in the gas <u>isothermally</u> heat energy must be transferred to the gas from an external source during the expansion in order to maintain its temperature constant. The expand would then follow Boyle's law, pV = constant.

The gas would expand <u>adiabatically</u> if no heat energy transfer, to or from the gas, occurs during the expansion, the external work done in pushing the piston forward being entirely at the expense of the stored up heat energy. Therefore the temperature of the gas will fall during the expansion. As for adiabatic compression the law for adiabatic expansion is pV^{δ} = constant.

During polytropic expansion, a partial amount of heat energy will be transferred to the gas from an outside source but not sufficient to maintain a uniform temperature during the expansion. The law for polytropic expansion is pV^n = constant as it is for polytropic compression.

With reference to *Figures 1 and 2* note that the adiabatic curve is the steepest, the isothermal curve is the least steep, and the polytropic curve lies between the two. Thus, the higher the index of the law of expansion or compression, the steeper will be the curve. It must also be noted that for any mode of expansion or compression in a closed system, the combination of Boyle's and Charles' laws, and the characteristic gas equation, are always true

$$\frac{pV}{T} = \text{constant} \qquad \therefore \frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$pV = mRT$$

Example 1

0.25m³ of air at 90 kN/m² and 10°C are compressed in an engine cylinder to a volume of 0.05m³, the law of compression being $pV^{1.4}$ = constant. Calculate (i) the final pressure, (ii) the final temperature, (iii) the mass of air in the cylinder, taking the characteristic gas constant for air R = 0.287 kJ/kg/K.

$$p_1V_1^{1.4} = p_2V_2^{1.4}$$

$$90 \times 0.25^{1.4} = p_2 \times 0.05^{1.4}$$

$$p_2 = \frac{90 \times 0.25^{1.4}}{0.05^{1.4}}$$

$$= 90 \times 5^{1.4} = \underline{856.7 \text{ kN/m}^2} \text{ **Ans (i)**}$$

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{90 \times 0.25}{283} = \frac{856.7 \times 0.05}{T_2}$$

$$T_2 = \underline{283 \times 856.7 \times 0.05} = 538.8\text{K}$$

$$= \underline{265.8°\text{C}} \text{ **Ans (ii)**}$$

$$p_1V_1 = mRT_1$$

$$m = \frac{90 \times 0.25}{0.287 \times 283} = \underline{0.277 \text{ kg}} \text{ **Ans (iii)**}$$

Example 2

0.07m³ of gas at 4.14 MN/m² is expanded in an engine cylinder and the pressure at the end of expansion is 310 kN/m². If expansion follows the law $pV^{1.35}$ = constant, find the final volume.

$$p_1V_1^{1.35} = p_2V_2^{1.35}$$

$$4140 \times 0.07^{1.35} = 310 \times V_2^{1.35}$$

$$V_2 = 0.07 \times \sqrt[1.35]{\frac{4140}{310}} = \underline{0.4774m^3} \text{ Ans}$$

Example 3

0.014m³ of gas at 3.15 MN/m² is expanded in a closed system to a volume of 0.154m³ and the final pressure is 120 kN/m². If the expansion takes place according to the law pV^n = constant, find the value of n

$$p_1V_1^n = p_2V_2^n$$

$$3150 \times 0.014^n = 120 \times 0154^n$$

$$\frac{3150}{120} = \left\{\frac{0.154}{0.014}\right\}^n$$

$$26.25 = 11^n$$

$$\underline{n = 1.363} \text{ Ans}$$

Determination of n from Graph

It is quite difficult to obtain two pairs of sufficiently accurate values of pressure and volume from a running engine to enable the law of expansion or compression to be determined as in the previous example. One practical method of finding a fairly close approximation of the law is as follows:

(i) measure a series of connected values of p and V from the curve
(ii) reduce the equation $pV^n = C$ to a straight line logarithmic equation
(iii) draw a straight line graph as near as possible through the plotted points of log p and log V to eliminate slight errors of measurement
(iv) determine the law of this graph to obtain the value of n, thus:

$$p \times V^n = C$$
$$\log p + n \log V = \log C$$
$$\log p = \log C - n \log V$$

This is the same form as equation $y = a - bx$ which represents a straight line graph.

The terms log p and log V are the two variables comparable with y and x respectively, and log C and n are constants comparable with a and b respectively. The constant n (like constant b) represents the slope of the straight line graph and, being a negative value, the line will slope downwards from left to right.

Example 4

The following related values of the pressure p in the kN/m² and the volume V in m³ were measured from the compression curve of an internal combustion engine indicator diagram. Assuming that p and V are connected by the law pVn = C, find the value of n.

p	3450	2350	1725	680	270	130
V	0.0085	0.0113	0.0142	0.0283	0.0566	0.0991

The scales of the graph can be of any convenient choice. In this case both pressure and volume can be expressed in more convenient units, the pressure in bars (1 bar = 10^5 N/m² = 10² kN/m²) to proportionally reduce the high figures, and the volume in litres (10³ litres = 1m³)

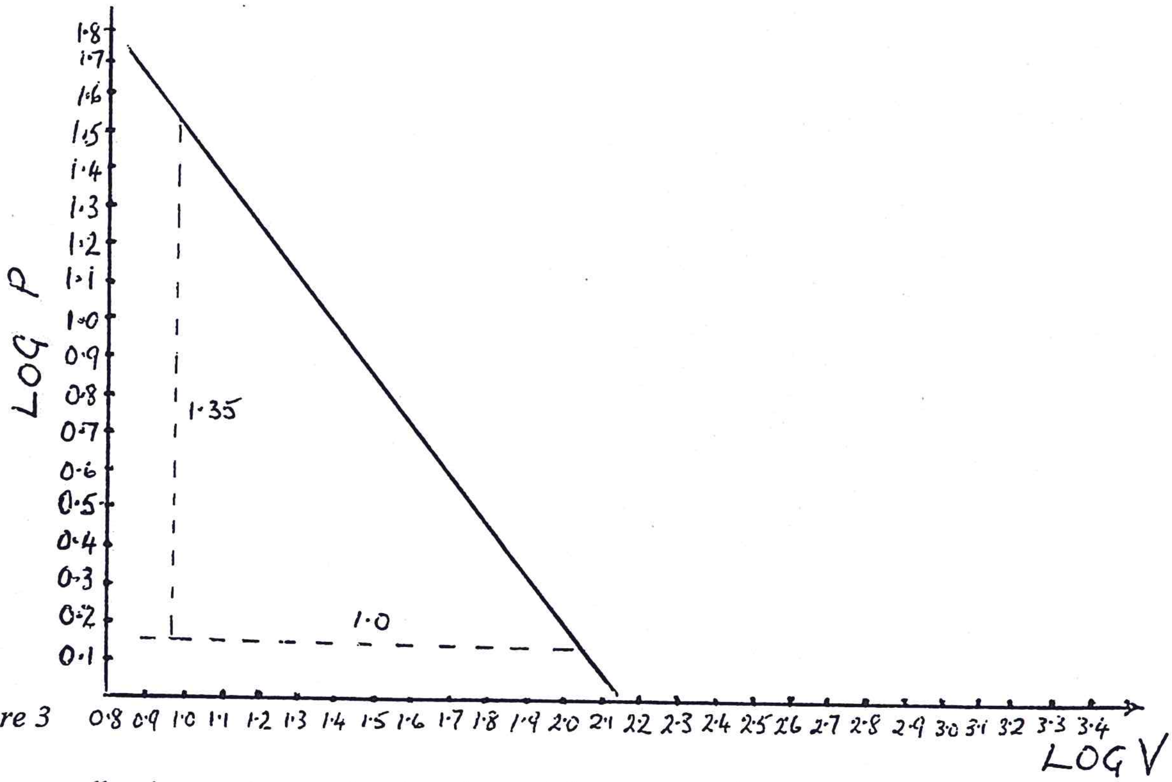

Figure 3

to express all volumes above unity, thereby avoiding negative loss and so reducing labour. Hence the following tabulated values of p and V are in bars and litres respectively with their corresponding logarithms to obtain graph plotting points from the respective pairs of log p and V (base 10 used here).

p (bar)	log p	V[litre]	log V
34.5	1.5323.5	8.5	0.9294
23.5	1.3711	11.3	1.0531
17.25	1.2368	14.2	1.1523
6.8	0.8325	28.3	1.4518
2.7	0.4315	56.6	1.7528
1.3	0.1139	99.1	1.9961

The graph is then plotted as shown in *Figure 3*. Note that it is not necessary to commence at zero origin when only the slope of the line (value of n) is required, a larger graph on the available squared paper can be drawn by starting and finishing to suit the minimum and maximum values to be plotted.

Choosing any two points on the line such as those shown

$$n = \frac{\text{decrease of log p}}{\text{decrease of log V}}$$

$$= \frac{0.15 \quad - \quad 0.15}{1.95 \quad - \quad 0.95}$$

$$= \frac{1.35}{1} = 1.35$$

$$\log p = \log C - 1.35 \log V$$

$$p = C \times V^{-1.35}$$

$$pV^{1.35} = \underline{\mathbf{C}}\ \textbf{Ans}$$

Ratios of Expansion and Compression

The ratio of expansion of gas in a cylinder is the ratio of the volume at the end of expansion to the volume at the beginning of expansion. It is usually denoted by r

$$\text{Ratios of expansion} = r = \frac{\text{final volume}}{\text{initial volume}}$$

It will be seen that in each of the above ratios, it is the larger volume divided by the smaller, therefore the ratio of expansion and ratio of compression is always greater than unity.

Relationships between Temperature and Volume, and Temperature and Pressure, When $pV^n = C$

As state previously, the equations

$$p_1V_1^n = p_2V_2^n \text{ and } \frac{p_1V_1}{T_2} = \frac{p_2V_2}{T_2}$$

are always true for any kind of expansion or compression of a perfect gas in a closed system. Some problems arise, however, where neither p_1 nor p_2 are given, and the unknown temperature or volume has to be solved by substituting the value of one of the pressures from one equation into the other. Similarly, where neither volume is given, substitution has to be made for one of the volumes to obtain the unknown temperature or pressure.

Substitution can be made in one of the above equations to eliminate either pressure or volume and so derive relationships for direct solution, as follows

$$p_1V_1^n = p_2V_2^n \therefore p_1 = \frac{p_2V_2^n}{V_1^n}$$

Substituting this value of p_1 into the combined law equation:

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{p_2 \times V_2^n \times V_1}{T_1 \times V_1^n} = \frac{p_2V_2}{T_2}$$

$$T_1 \times V_1^n \times p_2 \times V_2 = T_2 \times p_2 \times V_2^n \times V_1$$

$$\frac{T_1}{T_2} = \frac{p_2 \times V_2^n \times V_1}{p_2 \times V_2 \times V_1^n}$$

p2 cancels

dividing V_2^n by $V_2 = V_2^n \div V_2 = V_2^{n-1}$
dividing V_1^n by $V_1 = V_1^n \div V_1 = V_1^{n-1}$

$$\frac{T_1}{T_2} = \frac{V_2^{n-1}}{V_1^{n-1}}$$

$$\frac{T_1}{T_2} = \left\{\frac{V_2}{V_1}\right\}^{n-1} \qquad \text{(i)}$$

again from $p_1V_1^n = p_2V_2^n$ $\quad V_1^n = \dfrac{p_2V_2^n}{p_1}$

$$\therefore V_1 = p_2^{1/n} \times \frac{V_2}{p_1^{1/n}}$$

Substituting this value of V_1 into the combined law equation

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{p_1 \times p_2^{1/n} \times V_2}{T_1 \times p_1^{1/n}} = \frac{p_2 \times V_2}{T_2}$$

$$T_1 \times p_1^{1/n} \times p_2 \times V_2 = T_2 \times p_1 \times p_2^{1/n} \times V_2$$

$$\frac{T_1}{T_2} = \frac{p_1 \times p_2^{1/n} \times V_2}{p_1^{1/n} \times p_2 \times V_2}$$

V_2 cancels

dividing p_1 by $p_1^{1/n} = p_1 \div p_1^{1/n} = p_1^{1-1/n}$
dividing p_2 by $p_2^{1/n} = p_2 \div p_2^{1/n} = p_2^{1-1/n}$

$$\frac{T_1}{T_2} = \frac{p_1^{1-1/n}}{p_2^{1-1/n}}$$

$$\frac{T_1}{T_2} = \frac{\{p_1\}^{1-1/n}}{\{p_2\}^{1-1/n}}$$

$$\frac{T_1}{T_2} = \left\{\frac{p_1}{p_2}\right\}^{n-1/n} \qquad \text{(ii)}$$

From (i) and (ii) we have the very useful relationship

$$\frac{T_1}{T_2} = \left\{\frac{V_2}{V_1}\right\}^{n-1} = \left\{\frac{p_1}{p_2}\right\}^{n-1/n}$$

For an adiabatic process, the adiabatic index δ is substituted for the polytropic index n.

Figure 4 shows the pressure volume diagram representing work done at constant pressure. The graph is a straight horizontal line and the area under it is a rectangle. The area of a rectangle is height x length which, in this case, is $p \times (V_2 - V_1)$. Hence the area under the pressure volume line represents work done.

Considering cases where the pressure falls during the expansion of gas. The formula giving the area under the polytropic curve representing the general relationship between pressure and volume, i.e. $pVn = C$ can only be derived satisfactorily by the used of calculus. The expression is illustrated in *Figure 4* and derived in the following example.

Example

A quantity of gas undergoes a non flow process from an initial pressure of p_1 and volume V_1 to a final pressure p_2 and volume V_2.

(a) Given that the area beneath the pV curve is given by:

$$\int_1^2 p\,dV$$

show that the total area beneath the pV curve may be written as:

$$\frac{p_2V_2 - p_1V_1}{1 - n}$$

(b) During the process a quantity of heat q per unit mass is transferred. Show that q is given by:

$$q = (C_v + {}^{R}/_{1-n})\ (T2 - T1)$$

$$pV^n = C$$

$$p = {}^{C}/_{V^n}$$

$$\text{Area} = \int_1^2 p\,dV$$

$$= C\int_{V_1}^{V_2} V^{-n}\,dV$$

$$= pV^n \left[\frac{V^{-n+1}}{-n+1}\right]_{V_1}^{V_2}$$

$$= \frac{pV^n\,(V_2^{1-n} - V_1^{1-n})}{1-n}$$

$$\text{Area} = \frac{p_2V_2 - p_1V_1}{1-n}$$ **Ans (a)**

Heat Transfer = Change of Internal Energy + External Work

$$q = C_v(T_2 - T_1) + \frac{p_2V_2 - p_1V_1}{1-n}$$

$$pv = RT$$

$$q = C_v\,(T_2 - T_1) + \frac{R(T_2 - T_1)}{1-n}$$

$$q = (C_v + {}^{R}/_{1-n})\,(T_2 - T_1)$$ **Ans (b)**

Work done during polytropic expansion $= \dfrac{p_1V_1 - p_2V_2}{n-1}$

This is the general expression for work done. For adiabatic expansion, n is replaced by δ. For isothermal expansion however, since the value of n is 1, and as $p1V1 = p2V2$ then substitution in this expression for work will produce $0 \div 0$ which is indeterminate. A different expression is therefore necessary to obtain work during isothermal expansion (refer to *Figure 4*).

$$pV = C$$

$$p = \frac{C}{V}$$

$$\text{Area} = \int_1^2 p\,dV$$

$$= C\int_{V_1}^{V_2} \frac{dV}{V}$$

$$= C(\ln V_2 - \ln V_1)$$

$$= pV \ln\left(\frac{V_2}{V_1}\right)$$

$$\text{Work done during isothermal expansion} = pV \ln r$$

where r is the ratio of expansion.

The above expressions give the work done by the gas during expansion. The same expressions give the work done <u>on</u> the gas during compression.

In the case of expansion the initial condition of p1V1 will exceed the final condition of p2V2 and the expression for work will produce a <u>positive</u> result, indicating that work is done <u>by</u> the gas in pushing the piston forward. Conversely, for compression, the initial condition of p1V1 will be less than the final condition of p2V2 and therefore a <u>negative</u> result will be obtained, indicating that work is done <u>on</u> the gas by the piston.

It is important to bear in mind that in calculating work, the units must be consistent. For example, to express work in kJ, the pressure must be in kN/m^2 and the volume in m^3, thus

$$kN/m^2 \times m^3 = kNm = kJ$$

Work is transfer of energy, therefore the above can be referred to as work transfer from the closed system within the boundary of the cylinder to the external mechanism, or vice-versa. In the first case, when the gas expands, work is being transferred from the energy in the gas to the piston, which, in turn, transmits the work transfer in this case is referred to as being positive. In the second case, when the gas is compressed, work is being transferred from the crankshaft, through the connecting mechanism and piston to the gas, thereby increasing the energy in the gas, and this work transfer is called negative.

Further, since $pV = mRT$, the expressions for work may be stated in terms of mRT instead of pV.

Polytropic expansion

$$\text{Work} = \frac{p_1V_1 - p_2V_2}{n-1} = \frac{mR\,(T_1 - T_2)}{n-1}$$

Isothermal expansion

$$\text{Work} = pV \ln r = mRT \ln r$$

Example

$0.04m^3$ of gas at a pressure of 1482 kN/m^2 is expanded isothermally until the volume is $0.09m^3$. Calculate the work done during the expansion.

$$\text{Work done} = pV \ln r$$

$$r = \frac{\text{final volume}}{\text{initial volume}} = \frac{0.09}{0.04} = 2.25$$

$$\text{Work done} = 1482 \times 0.04 \times 0.81809$$

$$= \underline{\mathbf{48.08\ kJ}} \quad \textbf{Ans}$$

Example

7.08 litres of air at a pressure of 13.79 bar and temperature 335°C are expanded according to the law $pV^{1.32}$ = constant, and the final pressure is 1.206 bar. Calculate (i) the volume at the end of expansion, (ii) the work transfer from the air, (iii) the temperature at the end of expansion, (iv) the mass of air in the system, taking R = 0.287 kJ/kg K.

$$p_1V_1^{1.32} = p_2V_2^{1.32}$$

$$1379 \times 0.00708^{1.32} = 120.6 \times V_2^{1.32}$$

$$V_2 = 0.00708 \times \sqrt[1.32]{1379/120.6}$$

$$= 0.04484\text{m}^3 \text{ or } \underline{44.84 \text{ litres}} \quad \textbf{Ans (I)}$$

Note that if the units of pressure and volume are of the same kind on each side of the equation, the units cancel each other out and hence any convenient units can be used. The above could therefore be worked in bars of pressure and litres of volume in which the question data is given. However, as pointed out, it is essential to work in fundamental units in such expressions as used in parts (ii) and (iv) of this problem, it is preferential to use fundamental units throughout by expressing the pressure in kN/m² and the volume in m³.

$$\text{Work} = \frac{p_1V_1 - p_2V_2}{n-1}$$

$$= \frac{1379 \times 0.00708 - 120.6 \times 0.04884}{1.32 - 1} = \underline{13.61 \text{ kJ}} \quad \textbf{Ans (ii)}$$

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{1379 \times 0.00708}{608} = \frac{120.6 \times 0.04884}{T_2}$$

$$T_2 = \frac{608 \times 120.6 \times 0.04884}{1379 \times 0.00708}$$

$$T_2 = 336.6\text{K} = \underline{63.6°\text{C}} \quad \textbf{Ans (iii)}$$

$$p1V1 = mRT$$

$$m = \frac{1379 \times 0.00708}{0.287 \times 608} = \underline{0.05595 \text{ kg}} \quad \textbf{Ans (iv)}$$

Example

A perfect gas is compressed in a cylinder according to the law $pV^{1.3}$ = constant. The initial condition of the gas is 1.05 bar, 0.34m³ and 17°C. If the final pressure is 6.32 bar, calculate (i) the mass of gas in the cylinder, (ii) the final volume, (iii) the final temperature, (iv) the work done to compress the gas, (v) the change in internal energy, (vi) the transfer of heat between the gas and cylinder walls.
Take Cv = 0.7175 kJ/kgK and R = 0.287 kJ/kgK.

$$p_1V_1 = mRT$$

$$m = \frac{105 \times 0.34}{0.287 \times 290}$$

$$= \underline{0.4289\text{ kg}} \quad \textbf{Ans (i)}$$

$$p_1V_1^{1.3} = p_2V_2^{1.3}$$

$$1.05 \times 0.34^{1.3} = 6.32 \times V_2^{1.3}$$

$$V_2 = 0.34 \times \sqrt[1.3]{1.05/6.32}$$

$$= \underline{0.08549\text{m}^3} \quad \textbf{Ans (ii)}$$

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{1.05 \times 0.34}{290} = \frac{6.32 \times 0.08549}{T_2}$$

$$T_2 = \frac{290 \times 6.32 \times 0.08549}{1.05 \times 0.34}$$

$$T_2 = 438.8\text{K}$$

$$= \underline{165.8°\text{C}} \quad \textbf{Ans (iii)}$$

Example

Air is expanded adiabatically from a pressure of 800 kN/m^2 to 128 kN/m^2. If the final temperature is 57°C, calculate the temperature at the beginning of expansion, taking $\delta = 1.4$.

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\delta-1}{\delta}}$$

$$\frac{T_1}{330} = \left(\frac{800}{128}\right)^{2/7}$$

$$T_1 = 330 \times 6.25^{2/7}$$

$$= 557.1K$$

$$= \underline{284.1°C} \text{ Ans}$$

Example

The ratio of compression in a petrol engine is 9 to 1. Find the temperature of the gas at the end of compression if the temperature at the beginning is 24°C, assuming compression to follow the law pV^n = constant where $n = 1.36$.

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{n-1} \quad \text{or} \quad \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{n-1}$$

$$T_2 = 297 \times 9^{0.36} = 655.1K$$

$$= \underline{382.1°C} \text{ Ans}$$

Example

The volume and temperature of a gas at the beginning of expansion are $0.0056m^3$ and 183°C, at the end of expansion the values are $0.0238m^3$ and 22°C respectively. Assuming expansion follows the law $pV^n = C$, find the value of n.

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{n-1}$$

$$\frac{456}{295} = \left(\frac{0.0238}{0.0056}\right)^{n-1}$$

$$1.546 = 4.25^{n-1}$$

$$n = \underline{1.301} \text{ Ans}$$